

On the Capacity of a Code Division Multiple Access System

Satish Babu Korada, Nicolas Macris

Abstract—We consider multiuser communication on a binary input additive white Gaussian noise channel using Randomly Spread-Code Division Multiple Access. We show concentration of various quantities of the system including the capacity and the free energy. We also obtain a tight upper bound on the capacity of this system in the large user limit which matches with the replica solution. The method is quite general and can be extended to many other multiuser scenarios.

I. INTRODUCTION

A. Motivation

There is a natural connection between various communications systems and statistical mechanics of random spin systems, stemming from the fact that in both systems there is a large number of interacting degrees of freedom. So far, there have been applications of two important but somewhat complementary approaches of statistical mechanics.

The first one is the very important but mathematically uncontrolled *replica trick* or *cavity method* [1]. The merit of this approach is to obtain conjectural but rather explicit formulas for quantities of interest such as the free energy, the conditional entropy, error probability etc. In some cases the natural fixed point structure embodied in the formulas even allows to guess good iterative algorithms. This program has been carried out for linear error correcting codes [13], source coding [14], multi-user settings like broadcast channel, multiple access channel, and also for the case of interest here, namely, communication with a Code Division Multiple Access scheme (CDMA) [2].

The second type of approach aims at a rigorous proof of the replica formulas and is sometimes referred to as the *interpolation method* [4]. The basic idea is to study a measure which interpolates between the true (posterior) measure at hand and the mean field measure underlying the replica formula. This program has been successful in a few situations mainly limited to communication over binary symmetric channels with linear error correcting codes on sparse graphs (LDPC codes) [3], [5].

In the present contribution we address the largely open question of the interpolation method for a non-linear system with dense underlying graph like the CDMA system.

B. Communication Setup

We consider a situation where K users send binary information symbols $\underline{x} = (x_1, \dots, x_K)$, $x_k = \pm 1$ to a common receiver, through a single AWGN channel. Each

user has a signature sequence $\underline{s}_k = (s_{1k}, \dots, s_{Nk})^t$ assumed to be known to the receiver. At each time interval $i = 1, \dots, N$ the received signal is $\frac{1}{\sqrt{N}} \sum_{k=1}^K s_{ik} x_k + n_i$ where $\underline{n} = (n_1, \dots, n_N)^t$ are independent identically distributed Gaussian variables $\mathcal{N}(0, \sigma^2)$ and σ is the noise amplitude. The scaling factor $1/\sqrt{N}$ is introduced so that the energy of each user per information bit is normalized to 1. Here we take s_{ik} as generated randomly from independent identically distributed standard Gaussians $\mathcal{N}(0, 1)$ and denote the corresponding $N \times K$ random matrix as \mathbf{S} and a realization of the matrix with \mathbf{s} . The received vector $\underline{y} = (y_1, \dots, y_N)^t$ can be expressed as

$$\underline{y} = \frac{1}{\sqrt{N}} \mathbf{S} \underline{x} + \underline{n}$$

If all users communicate at the same rate R the maximum achievable such rate is given by

$$C(\mathbf{s}) = \frac{1}{K} \max_{p(\underline{x})} I(\underline{X}; \underline{Y} | \mathbf{S} = \mathbf{s})$$

Our main interest is to compute this quantity in the large system limit $K, N \rightarrow \infty$ with $\frac{K}{N} = \beta$ fixed. We can show that in the large system limit $I(\underline{X}; \underline{Y} | \mathbf{S})$ concentrates on its expectation over \mathbf{S} uniformly in $p(\underline{x})$. The latter quantity is maximized for a uniform input distribution $p(\underline{x}) = 2^{-K}$, so that from general arguments there is no loss of generality in considering

$$C(\mathbf{s}) = \ln 2 - \frac{1}{K} H(\underline{X} | \underline{Y}, \mathbf{S} = \mathbf{s}) \quad (1)$$

The conditional entropy in (1) is the average (over \underline{Y} only) Shannon entropy of the posterior distribution

$$p(\underline{x} | \underline{y}, \mathbf{s}) = \frac{1}{Z(\underline{y}, \mathbf{s})} \exp\left(-\frac{1}{2\sigma^2} \|N^{-\frac{1}{2}} \mathbf{s} \underline{x} - \underline{y}\|^2\right) \quad (2)$$

with the normalization factor

$$Z(\underline{y}, \mathbf{s}) = \sum_{\underline{x}} \exp\left(-\frac{1}{2\sigma^2} \|N^{-\frac{1}{2}} \mathbf{s} \underline{x} - \underline{y}\|^2\right)$$

The average over \underline{Y} is carried out with the distribution

$$p(\underline{y} | \mathbf{s}) = \sum_{\underline{x}^0} p(\underline{y} | \underline{x}^0, \mathbf{s}) p(\underline{x}^0)$$

where \underline{x}^0 is to be interpreted as an input signal.

C. Tanaka's Formula

Tanaka [2] computed

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\mathbf{s}}[C(\mathbf{S})] = \ln 2 - \lim_{K \rightarrow \infty} \frac{1}{K} \mathbb{E}_{\mathbf{s}}[H(\underline{X} | \underline{Y}, \mathbf{S})]$$

The authors are affiliated to the School of Information and Communication Sciences, Ecole Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland { satish.korada, nicolas.macris }@epfl.ch

where $\mathbb{E}_{\mathbf{S}}[\cdot]$ is expectation is over the signature sequences, by using the replica trick. The formula he obtained is

$$\lim_{K \rightarrow \infty} \frac{1}{K} \mathbb{E}_{\mathbf{S}}[H(\underline{X}|\underline{Y}, \mathbf{S})] = \max_{m \in [0,1]} h_{RS}(m)$$

where h_{RS} is the replica symmetric entropy functional given by

$$h_{RS}(m) = \int Dz \ln(2 \cosh(\sqrt{\lambda}z + \lambda)) - \frac{\lambda}{2}(1+m) - \frac{1}{2\beta} \ln \left(1 + \frac{\beta}{\sigma^2}(1-m) \right) \quad (3)$$

with

$$\lambda = \frac{1}{\sigma^2 + \beta(1-m)}$$

Here Dz is the standard gaussian measure $Dz \equiv \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$. It is easy to see that the maximizer must satisfy the fixed point condition

$$m = \int Dz \tanh(\sqrt{\lambda}z + \lambda) = \int Dz \tanh^2(\sqrt{\lambda}z + \lambda) \quad (4)$$

Recently Montanari and Tse [6] have reported some progress towards a proof of this formula for $\beta \leq \beta_s$ where β_s is the maximal value of β such that the solution of (4) remains unique. The authors first solve the case of sparse signature sequence (using the area theorem and the data processing inequality) in the limit $K \rightarrow \infty$. Then the dense signature sequence (which is of interest here) is recovered by exchanging the $K \rightarrow \infty$ and *sparse* \rightarrow *dense* limits.

Let us also note at this point that for Gaussian inputs there is an analogous *replica formula* which can be obtained rigorously through the application of random matrix theory (RMT) [7].

D. New Results

Our contribution here is two fold.

Firstly we provide the necessary concentration theorem on $C(\mathbf{S})$ which ensures that it is sufficient to consider $\mathbb{E}_{\mathbf{S}}[C(\mathbf{S})]$. For gaussian inputs known general concentration results from RMT should suffice. However for binary inputs, it turns out that the whole mathematical underpinning of the problem is very different. In [8] we treated the case of gaussian spreading sequence by using powerful probabilistic tools developed for Lipschitz functions of many gaussian random variables (this is briefly reported in section II). Here we give a proof of concentration in the case of binary spreading sequence using ideas from spin-glass theory due to Pastur, Shcherbina and Tirozzi [9], [10]. This is the subject of section VI.

Secondly, in the case of gaussian spreading sequences and binary inputs, we prove that Tanaka's formula (3) is an upper bound to the capacity for all values of β , namely

$$\lim_{K \rightarrow \infty} \mathbb{E}[C(\mathbf{S})] \leq \ln 2 - \max_{m \in [0,1]} h_{RS}(m)$$

The heart of the proof, in section III, is the interpolation method but, as will be seen we also need to prove

concentration theorems for the empirical average of the “magnetization”, $m_1 = \frac{1}{K} \sum_{k=1}^K x_k$. We feel that this later result has its own intrinsic interest and is presented in section IV. The proof uses a crucial feature that is specific to the present communication set up, namely the channel symmetry, which induces a gauge symmetry for the associated spin-glass.

Let us note that there are potentially many extensions and applications of the present methods to other settings. Two of them which are briefly sketched here are the case of unequal power, and that of gaussian inputs (for which we already have RMT results). Other problems that will be reported in a more detailed work concern extensions to communication set-ups such as CDMA with (LDPC) coded inputs, MIMO communication with binary inputs, and colored noise.

A open problem which we cannot yet treat with the present methods is to prove the above bound for binary spreading sequences. The replica method leads to the same formula as (3) and we think that the interpolation methods can be adapted to this case also (see comments in section V).

II. CONCENTRATION OF CAPACITY AND FREE ENERGY

For completeness, and since we will need some of them, we first state without proof the results for the case of $s_{ik} \sim \mathcal{N}(0, 1)$, which have appeared in [8]. If the spreading sequences are drawn from a gaussian distribution, we can use the Lipschitz property of the capacity and free energy functions and derive the following results.

Theorem 1: [concentration of the capacity.] Given any $\epsilon > 0$, there exists an integer $K_1 = O(|\ln \epsilon|)$ and a strictly positive constant α_1 such that for all $K > K_1$,

$$\mathbb{P}[|C(\mathbf{S}) - \mathbb{E}_{\mathbf{S}}[C(\mathbf{S})]| \geq \epsilon] \leq 3e^{-\alpha_1 \epsilon^2 K}$$

One can take $\alpha_1 = \frac{1}{16} \sigma^4 (64\beta + 32 + \sigma^2)^{-1}$.

We also prove the more general concentration result for the “free energy”. The free energy is defined for each realization of the channel output and spreading sequence by

$$f(\underline{y}, \mathbf{s}) = \frac{1}{K} \ln Z(\underline{y}, \mathbf{s})$$

and is related to the capacity by the following formula (see [8])

$$C(\mathbf{s}) = \ln 2 - \frac{1}{2\beta} - \mathbb{E}_{\underline{Y}|\mathbf{s}=\mathbf{s}}[f(\underline{Y}, \mathbf{s})] \quad (5)$$

We have

Theorem 2: [concentration of free energy.] Given any $\epsilon > 0$, there exists an integer $K_2 = O(|\ln \epsilon|)$ and a strictly positive constant α_2 such that for all $K \geq K_2$,

$$\mathbb{P}[K^{-1} |\ln Z(\underline{Y}, \mathbf{S}) - \mathbb{E}_{\underline{Y}, \mathbf{S}}[\ln Z(\underline{Y}, \mathbf{S})]| \geq \epsilon] \leq 3e^{-\alpha_2 \epsilon^2 \sqrt{K}}$$

One can take $\alpha_2 = \frac{1}{32} \sigma^4 \beta^{3/2} (2\sqrt{\beta} + \sigma)^{-2}$.

III. TIGHT UPPER BOUND ON CAPACITY

The integral term in (3) suggests that we can replace the original system with a simpler system where the user bits are sent through K independent Gaussian channels given by

$$\tilde{y}_k = x_k + \frac{1}{\sqrt{\lambda}} w_k$$

where $w_k \sim \mathcal{N}(0, 1)$ and $\lambda = (\sigma^2 + \beta(1 - m))^{-1}$ is an effective SNR where m is given by the fixed point equation (4). Of course this argument is not exactly true because this effective system would not account for the extra terms in (3), but has the merit of identifying the correct interpolation.

The original CDMA system corresponds to a symmetric channel. And moreover since the spreading sequences are symmetric, for the computation of $\mathbb{E}[C(\mathbf{S})]$ we can assume that the all one sequence is transmitted, and set $x_k^0 = 1$ without loss of generality. Note that this assumption is not true for a particular realization of \mathbf{S} .

We introduce an interpolating parameter $t \in [0, 1]$ such that the independent gaussian channels correspond to $t = 0$ and the original CDMA system corresponds to $t = 1$. More precisely the interpolating “communication system” has a posterior distribution

$$p_t(\underline{x}|\underline{y}, \mathbf{s}) = \frac{\exp(H_t(\underline{x}) + h_u(\underline{x}))}{Z_t} \quad (6)$$

with

$$Z_t = \sum_{\underline{x}} \exp(H_t(\underline{x}) + h_u(\underline{x}))$$

In this distribution there are two “hamiltonians”, the proper “interpolating hamiltonian” $H_t(\underline{x})$ and a “small perturbation” $h_u(\underline{x})$ that is introduced for technical reasons that will become clear later.

Let us first describe the interpolating hamiltonian. We set $z_k = x_k^0 - x_k = 1 - x_k$

$$H_t(\underline{x}) = -\frac{B(t)}{2} \sum_{i=1}^N \left(\frac{n_i}{\sqrt{B(t)}} + \frac{1}{\sqrt{N}} \sum_{k=1}^K s_{ik} z_k \right)^2 - \frac{\lambda(t)}{2} \sum_{k=1}^K \left(\frac{w_k}{\sqrt{\lambda(t)}} + z_k \right)^2$$

Here the independent noise variables n_i and w_k are standard gaussians. This interpolating hamiltonian corresponds to a “mixed” channel as shown in the following figure.

If we denote the SNR of the original gaussian channel as $B (= \frac{1}{\sigma^2})$, then the integral term suggests that the effective SNR seen by each user is $\frac{B}{B + \beta(1 - m)}$. Therefore, in the interpolating system the effective SNR seen by each user in the CDMA part is $\frac{B(t)}{B(t) + \beta(1 - m)}$ and in the independent channel is $\lambda(t)$. The interpolating functions $B(t)$ and $\lambda(t)$ are chosen such that the effective total SNR is fixed

$$\frac{B(t)}{1 + \beta B(t)(1 - m)} + \lambda(t) = \frac{B}{1 + \beta B(1 - m)} \quad (7)$$

and the following boundary values are satisfied $B(0) = 0$, $B(1) = \sigma^{-2}$ and $\lambda(0) = \lambda$, $\lambda(1) = 0$. The parameter m is to

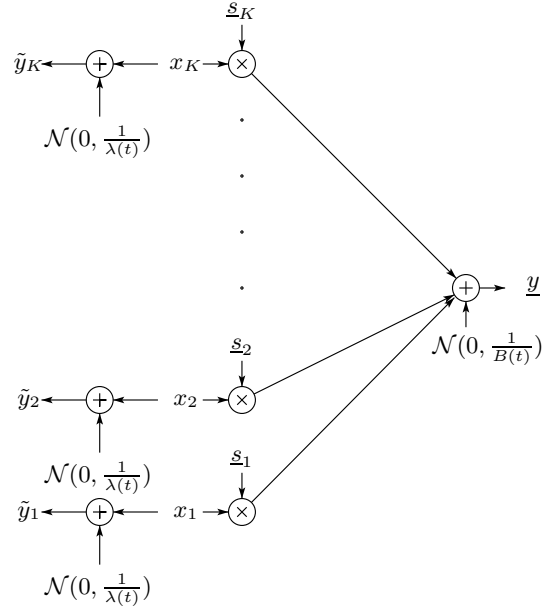


Fig. 1. The information bits x_k are transmitted through the normal CDMA channel with variance $\frac{1}{B(t)}$ and individual Gaussian channels with noise $\frac{1}{\lambda(t)}$

be considered as fixed to any arbitrary value in $[0, 1]$. All the subsequent calculations are independent of its value, which is to be optimized at the very end. There is a whole class of interpolating functions satisfying the above conditions and we do not need to specify them more precisely except for the fact that we need $B(t)$ increasing, $\lambda(t)$ decreasing and with continuous first derivatives.

The perturbation hamiltonian is carefully tuned to

$$h_u(\underline{x}) = \sqrt{u} \sum_{k=1}^K h_k x_k + u \sum_{k=1}^K x_k - \sqrt{u} \sum_{k=1}^K |h_k| \quad (8)$$

where h_k are i.i.d. $h_k \sim \mathcal{N}(0, 1)$. Our results below are valid for general values of $u > 0$, but in the present application we will have $u \rightarrow 0$.

At this point let us note that to ease the notation all subsequent expectations \mathbb{E} are taken with respect to n_i , w_k , s_{ik} and h_k .

We introduce the average free energy

$$\mathcal{F}(t) = \frac{1}{K} \mathbb{E}[\ln Z_t] \quad (9)$$

From the definitions above it is not difficult to see that

$$\mathcal{F}(1) = -\frac{1}{2} + \frac{1}{K} \mathbb{E}[\ln Z(\underline{y}, \mathbf{s})] + O(\sqrt{u})$$

and is therefore directly related to the capacity of the original CDMA system. It will be important to keep in mind that the $O(\sqrt{u})$ is uniform in the system size. On the other hand $\mathcal{F}(0)$ corresponds to independent users and is easy to compute

$$\mathcal{F}(0) = -\frac{1}{2\beta} - \lambda - \frac{1}{2} + \int Dz \ln(2 \cosh(\sqrt{\lambda} z + \lambda)) + O(\sqrt{u})$$

Because of the fundamental theorem of calculus

$$\mathcal{F}(1) = \mathcal{F}(0) + \int_0^1 \frac{d\mathcal{F}(t)}{dt} dt$$

our task is reduced to computing the t -derivative of (9). Before presenting this computation let us explain its result.

Let the angle bracket $\langle - \rangle_t$ denote the average with respect to the Gibbs measure (6) and $m_1 = \frac{1}{K} \sum_{k=1}^K x_k$ denote the “magnetization”. The average of the later with respect to the Gibbs measure

$$\langle x_k \rangle_t = \sum_{\underline{x}} x_k p_t(\underline{x} | \underline{y}, \underline{\tilde{y}}, \mathbf{s})$$

can be interpreted as a soft bit MAP estimate (pertaining to the interpolating communication channel). To lighten the notation, we do not show the u dependence of the averages.

A closely related quantity is the “overlap parameter” $q_{12} = \frac{1}{K} \sum_{k=1}^K x_k^{(1)} x_k^{(2)}$ where $x_k^{(1)}$ and $x_k^{(2)}$ are independent copies (“replicas”) of the x_k . This means that their joint distribution is $p_t(\underline{x}^{(1)} | \underline{y}, \underline{\tilde{y}}, \mathbf{s}) p_t(\underline{x}^{(2)} | \underline{y}, \underline{\tilde{y}}, \mathbf{s})$. The Gibbs average $\langle q_{12} \rangle_t$ with respect to this joint distribution is denoted (by a slight abuse of notation) with the same bracket $\langle - \rangle_t$. The important thing to notice is that the replicas are “coupled” through the common randomness of noises n_i , w_k and h_k .

There are two crucial ingredients that enter in our computation of the t -derivative. One is the gauge symmetry (related to channel symmetry) leading to so called Nishimori identities. In appendix A we discuss the necessary identities, but the most important one is perhaps that the full probability distributions (i.e with respect to noise expectations $\mathbb{E}[-]$ and Gibbs measure $\langle - \rangle_t$) are equal (see lemma 7 in appendix A). This implies in particular that

$$\mathbb{E}[\langle m_1 \rangle_t] = \mathbb{E}[\langle q_{12} \rangle_t] \quad (10)$$

The second one is that m_1 (and q_{12}) concentrate, namely

Theorem 3: Fix any $\epsilon > 0$. For a.e. $u > \epsilon$,

$$\lim_{N \rightarrow \infty} \int_0^1 \mathbb{E}[\langle m_1 - \mathbb{E}\langle m_1 \rangle_t \rangle_t] dt = 0$$

The proof of this theorem, which makes use of the careful tuning of the perturbation hamiltonian, has an interest of its own and is presented section IV. Note that the Nishimori identity together with this concentration phenomenon “explains” why the replica symmetry is not broken and Tanaka’s formula holds.

We are now ready to state the result of the computation of the t -derivative,

$$\begin{aligned} \mathcal{F}(1) = \mathcal{F}(0) - \frac{\lambda}{2} m - \frac{1}{2\beta} \ln(1 + \beta B(1 - m)) \\ + \int_0^1 R(t) dt + o_N(1) \end{aligned} \quad (11)$$

where for a.e $u > \epsilon$, $\lim_{N \rightarrow \infty} o_N(1) = 0$, and the remainder term is

$$R(t) = \frac{\beta B'(t) B(t) (\mathbb{E}\langle m_1 - m \rangle_t)^2}{2(1 + \beta B(t)(1 - m))^2 (1 + \beta B(t) \mathbb{E}\langle 1 - m_1 \rangle_t)}$$

By explicit inspection of this formula we see that $R(t) \geq 0$ for all m . Thus taking first the limit $N \rightarrow \infty$, then $u \rightarrow \epsilon$ (along some appropriate sequence) and finally $\epsilon \rightarrow 0$, we obtain the main theorem of this work,

Remark 1: To obtain the formula for $R(t)$ we have to use theorem 3 which is true only for a.e. $u > 0$. This is the technical reason why we cannot work directly with $u = 0$, which corresponds to the original CDMA system.

Theorem 4: Consider the CDMA system with binary inputs, gaussian spreading sequence and gaussian noise, as defined in the introduction. We have

$$\mathbb{E}[C(\mathbf{s})] \leq \ln 2 - \max_{m \in [0,1]} h_{RS}(m)$$

where $h_{RS}(m)$ is given by the formulas (3).

Proof: In the rest of this section we carry out the calculation leading to (11). We will use repeatedly the integration by parts formula $\mathbb{E}[gh(g)] = v^2 \mathbb{E}[f'(g)]$ for a gaussian random variable $g \sim \mathcal{N}(0, v^2)$, where g plays the role of n_i , w_k or s_{ik} . This is why this proof is limited to gaussian spreading sequences.

Differentiating $\mathcal{F}(t)$ we get

$$\frac{d\mathcal{F}(t)}{dt} = T_1 + T_2 \quad (12)$$

with the following expressions for T_1 and T_2

$$\begin{aligned} T_1 = \frac{1}{K} \mathbb{E} \left\langle - \sum_i \left(n_i + \sqrt{\frac{B(t)}{N}} \sum_k s_{ik} z_k \right) \right. \\ \left. \times \frac{1}{\sqrt{N}} \frac{B'(t)}{2\sqrt{B(t)}} \sum_k s_{ik} z_k \right\rangle_t \end{aligned}$$

$$T_2 = - \frac{\lambda'(t)}{2\sqrt{\lambda(t)}K} \sum_k \mathbb{E}\langle w_k z_k \rangle_t - \frac{\lambda'(t)}{2K} \sum_k \mathbb{E}\langle z_k^2 \rangle_t$$

Let us first deal with the second term. Integration by parts with respect to w_k leads to

$$\begin{aligned} T_2 &= \frac{\lambda'(t)}{2\sqrt{\lambda(t)}K} \sum_k \mathbb{E}\langle (w_k + \sqrt{\lambda(t)} z_k) z_k \rangle_t \\ &- \frac{\lambda'(t)}{2\sqrt{\lambda(t)}K} \sum_k \mathbb{E}\langle z_k^{(1)} (w_k + \sqrt{\lambda(t)} z_k^{(2)}) \rangle_t \\ &- \frac{\lambda'(t)}{2K} \sum_k \mathbb{E}\langle z_k^2 \rangle_t \\ &= - \frac{\lambda'(t)}{2} \mathbb{E}\langle 1 - 2m_1 + q_{12} \rangle_t = - \frac{\lambda'(t)}{2} \mathbb{E}\langle 1 - m_1 \rangle_t \end{aligned}$$

From the relation between $\lambda(t)$ and $B(t)$ given in equation (7), T_2 can be rewritten in the form

$$T_2 = \frac{B'(t)}{2(1 + \beta B(t)(1 - m))^2} \mathbb{E}\langle 1 - m_1 \rangle_t \quad (13)$$

Now we deal with the more complicated term T_1 . Let $X_i = n_i + \frac{1}{\sqrt{N}} \sum_k s_{ik} z_k$ and $X_{11} = \frac{1}{N} \sum_i X_i^{(1)} X_i^{(1)}$ (note that this is the same as the sum of X_i^2). Then using some

extra Nishimori identity (see lemma 8 in appendix A), the term T_1 can be simplified as

$$\begin{aligned} T_1 &= \frac{B'(t)}{2\beta B(t)} \mathbb{E} \langle -X_{11} \rangle_t + \frac{B'(t)}{2\beta B(t)} \mathbb{E} \frac{1}{N} \sum_i n_i^2 \\ &\quad + \frac{B'(t)}{2\sqrt{B(t)}} \mathbb{E} \left\langle \frac{1}{K\sqrt{N}} \sum_{i,k} n_i s_{ik} z_k \right\rangle_t \\ &= \frac{B'(t)}{2\sqrt{B(t)}} \mathbb{E} \left\langle \frac{1}{K\sqrt{N}} \sum_{i,k} n_i s_{ik} z_k \right\rangle_t \end{aligned}$$

Now we use integration by parts with respect to s_{ik} ,

$$\begin{aligned} T_1 &= \frac{B'(t)}{2KN} \sum_{i,k} \mathbb{E} \langle n_i (-n_i - \sqrt{\frac{B(t)}{N}} \sum_l s_{il} z_l) z_k^2 \rangle_t + \\ &\quad \frac{B'(t)}{2KN} \sum_{i,k} \mathbb{E} \langle n_i (n_i + \sqrt{\frac{B(t)}{N}} \sum_l s_{il} z_l^{(2)}) z_k^{(1)} z_k^{(2)} \rangle_t \end{aligned}$$

and the Nishimori identity again in the form of lemma 9

$$\begin{aligned} T_1 &= \frac{B'(t)}{2KN} \sum_{i,k} \mathbb{E} \langle n_i (-n_i - \sqrt{\frac{B(t)}{N}} \sum_l s_{il} z_l) (1 - x_k) \rangle_t \\ &= -\frac{\beta B'(t)}{2} \mathbb{E} \frac{1}{N} \sum_i (n_i)^2 \left\langle \frac{1}{K} \sum_k (1 - x_k) \right\rangle_t \\ &\quad - \frac{\beta B'(t)}{2N} \mathbb{E} \left\langle \sqrt{\frac{B(t)}{N}} \sum_{i,l} n_i s_{il} z_l \frac{1}{K} \sum_k (1 - x_k) \right\rangle_t \end{aligned}$$

Since $\frac{1}{N} \sum_{i=1}^N n_i^2$ concentrates, we get

$$\begin{aligned} T_1 &= -\frac{B'(t)}{2} \beta \mathbb{E} \langle 1 - m_1 \rangle_t + o_N(1) \\ &\quad + \frac{\beta B'(t)}{2N} \mathbb{E} \left\langle \sqrt{\frac{B(t)}{N}} \sum_{i,l} n_i s_{il} z_l (-(1 - m_1)) \right\rangle_t \end{aligned}$$

At this point we use the following lemma that we will prove shortly

Lemma 1: The following holds

$$\begin{aligned} &\mathbb{E} \frac{1}{N} \left\langle \sqrt{\frac{1}{N}} \sum_{i,l} n_i s_{il} z_l (1 - m_1) \right\rangle_t \\ &= \mathbb{E} \frac{1}{N} \left\langle \sqrt{\frac{1}{N}} \sum_{i,l} n_i s_{il} z_l \right\rangle_t (1 - \mathbb{E} \langle m_1 \rangle_t) + o_N(1) \end{aligned}$$

with $\lim_{n \rightarrow \infty} o_N(1) = 0$ for almost every u .

Applying this lemma to the last expression for T_1 we obtain a closed affine equation for the later, whose solution is

$$T_1 = -\frac{B'(t) \mathbb{E} \langle 1 - m_1 \rangle_t}{2(1 + \beta B(t) \mathbb{E} \langle 1 - m_1 \rangle_t)} + o_N(1) \quad (14)$$

To complete the calculation leading to (11) we add and subtract the term $\frac{1}{2\beta} \ln(1 + \beta B(1 - m))$ from (12) and use the representation

$$\frac{1}{2\beta} \ln(1 + \beta B(1 - m)) = \frac{1}{2\beta} \int_0^1 \frac{\beta B'(t)(1 - m)}{1 + \beta B(t)(1 - m)} dt$$

together with expressions (13) and (14).

It remains to prove lemma 1. ■

Proof of lemma 1: By the Cauchy-Schwarz inequality

$$\begin{aligned} &\mathbb{E} \left\langle \frac{1}{N^{3/2}} \sum_{i,l} n_i s_{il} z_l (\mathbb{E} \langle m_1 \rangle_t - m_1) \right\rangle_t \\ &\leq \mathbb{E} \left\langle \left(\frac{1}{N^{3/2}} \sum_{i,l} n_i s_{il} z_l \right)^2 \right\rangle_t \mathbb{E} \langle (\mathbb{E} \langle m_1 \rangle_t - m_1)^2 \rangle_t \end{aligned}$$

Because of the concentration of the magnetization m_1 (theorem 3) it suffices to prove that

$$\mathbb{E} \left\langle \left(N^{-\frac{3}{2}} \sum_{i,l} n_i s_{il} z_l \right)^2 \right\rangle_t \leq C \quad (15)$$

for some constant C independent of N . The proof follows from the central limit theorem and is omitted here. ■

IV. CONCENTRATION OF MAGNETIZATION

The goal of this section is to prove theorem 3. The proof is organized in a succession of lemmas.

In the previous section we considered an average free energy $\mathcal{F}(t)$. A more basic object is the free energy itself $f(t) = \frac{1}{K} \ln Z_t$. In this section we need some notation to keep explicit track of the u -dependence of $f(t)$: we do the replacement $f(t) \rightarrow f(t, u)$ and $\mathcal{F}(t) \rightarrow \mathcal{F}(t, u)$. By the same methods used for theorem 2 we can prove

Lemma 2: There exists a strictly positive constant α (which remains positive for all t and u) such that

$$\mathbb{P}[|f(t, u) - \mathcal{F}(t, u)| \geq \epsilon] = O(e^{-\alpha \epsilon^2 \sqrt{K}})$$

The perturbation term (8) has been chosen carefully so that the following holds,

Lemma 3: When considered as a function of u , $f(t, u)$ is convex in u .

Proof: We simply evaluate the second derivative and show it is positive.

$$\frac{df(t, u)}{du} = \langle L(\underline{x}) \rangle_{t,u} - \frac{1}{2\sqrt{u}} \sum_k |h_k|$$

where we have defined

$$L(\underline{x}) = \frac{1}{K} \frac{1}{2\sqrt{u}} \sum_k h_k x_k + \frac{1}{K} \sum_k x_k$$

Differentiating again,

$$\begin{aligned} \frac{d^2 f(t, u)}{du^2} &= \frac{1}{K} \left\langle \frac{-1}{4u^{3/2}} \sum_k h_k x_k \right\rangle_{t,u} + \frac{1}{4u^{3/2} K} \sum_k |h_k| \\ &\quad + K (\langle L(\underline{x})^2 \rangle_{t,u} - \langle L(\underline{x}) \rangle_{t,u}^2) \geq 0 \end{aligned} \quad (16)$$

The quantity $L(\underline{x})$ turns out to be very useful and satisfies two concentration properties.

Lemma 4: For any $a > \epsilon > 0$ fixed,

$$\int_{\epsilon}^a du \mathbb{E} \left\langle \left| L(\underline{x}) - \langle L(\underline{x}) \rangle_{t,u} \right| \right\rangle_{t,u} = O\left(\frac{1}{\sqrt{K}}\right)$$

Proof: From equation (16), we have

$$\begin{aligned} \int_{\epsilon}^a du \mathbb{E} \left\langle \left(L(\underline{x}) - \langle L(\underline{x}) \rangle_{t,u} \right)^2 \right\rangle_{t,u} &\leq \int_{\epsilon}^a du \frac{1}{K} \frac{d^2}{du^2} \mathcal{F}(t, u) \\ &\leq \frac{1}{K} \left(\frac{d}{du} \mathcal{F}(t, a) - \frac{d}{du} \mathcal{F}(t, \epsilon) \right) = O\left(\frac{1}{K}\right) \end{aligned}$$

In the very last equality we use that the first derivative of \mathcal{F} is bounded for $u \geq \epsilon$. Using Cauchy-Schwarz inequality for \mathbb{E} we obtain the lemma. ■

Lemma 5: For any $a > \epsilon > 0$ fixed,

$$\int_{\epsilon}^a du \mathbb{E} \left| \langle L(\underline{x}) \rangle_{t,u} - \mathbb{E} \langle L(\underline{x}) \rangle_{t,u} \right| = O\left(\frac{1}{K^{\frac{1}{8}}}\right)$$

Proof: From convexity of $f(t, u)$ with respect to u (lemma 3) we have for any $\delta > 0$,

$$\begin{aligned} \frac{d}{du} f(t, u) - \frac{d}{du} \mathcal{F}(t, u) &\leq \frac{f(t, u + \delta) - f(t, u)}{u_{\delta}} - \frac{d}{du} \mathcal{F}(t, u) \\ &\leq \frac{f(t, u + \delta) - \mathcal{F}(t, u + \delta)}{\delta} - \frac{f(t, u) - \mathcal{F}(t, u)}{\delta} \\ &\quad + \frac{d}{du} \mathcal{F}(t, u + \delta) - \frac{d}{du} \mathcal{F}(t, u) \end{aligned}$$

A similar lower bound holds with δ replaced by $-\delta$. Now from lemma 2 we know that the first two terms are $O(K^{\frac{1}{4}})$. Thus from the formula for the first derivative in the proof of lemma 3 and the fact that the fluctuations of $\frac{1}{K} \sum_{k=1}^K |h_k|$ are $O(\frac{1}{\sqrt{K}})$ we get

$$\begin{aligned} \mathbb{E} \left| \langle L(\underline{x}) \rangle_{t,u} - \mathbb{E} \langle L(\underline{x}) \rangle_{t,u} \right| &\leq O\left(\frac{1}{\sqrt{K}}\right) + O\left(\frac{1}{K^{\frac{1}{4}}}\right) \\ &\quad + \frac{d}{du} \mathcal{F}(t, u + \delta) - \frac{d}{du} \mathcal{F}(t, u) \end{aligned}$$

We will choose $\delta = \frac{1}{K^{\frac{1}{8}}}$. Note that we cannot assume that the difference of the two derivatives is small because the first derivative of the free energy is not uniformly continuous in K (as $K \rightarrow \infty$ it may develop jumps at the phase transition points). The free energy itself is uniformly continuous (because of convexity). For this reason if we integrate with respect to u , we get

$$\int_{\epsilon}^a du \mathbb{E} \left| \langle L(\underline{x}) \rangle_{t,u} - \mathbb{E} \langle L(\underline{x}) \rangle_{t,u} \right| \leq O\left(\frac{1}{K^{\frac{1}{8}}}\right)$$

Using the two last lemmas we can prove theorem 3. ■

Proof of theorem 3: Combining the concentration lemmas we get

$$\int_{\epsilon}^a du \mathbb{E} \left| \langle L(\underline{x}) \rangle_{t,u} - \mathbb{E} \langle L(\underline{x}) \rangle_{t,u} \right| \leq O\left(\frac{1}{K^{\frac{1}{8}}}\right)$$

For any function $g(\underline{x})$ such that $|g(\underline{x})| \leq 1$, we have

$$\begin{aligned} \int_{\epsilon}^a du \mathbb{E} \left| \langle L(\underline{x}) g(\underline{x}) \rangle_{t,u} - \mathbb{E} \langle L(\underline{x}) \rangle_{t,u} \mathbb{E} \langle g(\underline{x}) \rangle_{t,u} \right| &\leq \int_{\epsilon}^a du \mathbb{E} \left| \langle L(\underline{x}) \rangle_{t,u} - \mathbb{E} \langle L(\underline{x}) \rangle_{t,u} \right| \\ &\leq \int_{\epsilon}^a du \mathbb{E} \left| \langle L(\underline{x}) \rangle_{t,u} - \mathbb{E} \langle L(\underline{x}) \rangle_{t,u} \right| \end{aligned}$$

More generally the same thing holds if one takes a function depending on many replicas such as $g(\underline{x}^{(1)}, \underline{x}^{(2)}) = q_{12}$. Using integration by parts formula with respect to h_k ,

$$\begin{aligned} \mathbb{E} \langle L(\underline{x}) q_{12} \rangle_{t,u} &= \mathbb{E} \left\langle \frac{1}{2N\sqrt{u}} \sum_k h_k x_k q_{12} \right\rangle_{t,u} + \mathbb{E} \langle m_1 q_{12} \rangle_{t,u} \\ &= \frac{1}{2} \mathbb{E} \langle (1 + q_{12}) q_{12} \rangle_{t,u} - \frac{1}{2} \mathbb{E} \langle (q_{13} + q_{14}) q_{12} \rangle_{t,u} \\ &\quad + \mathbb{E} \langle m_1 q_{12} \rangle_{t,u} \\ &= \frac{1}{2} \mathbb{E} \langle (1 + q_{12}) q_{12} \rangle_{t,u} = \frac{1}{2} \mathbb{E} \langle m_1 + m_1^2 \rangle_{t,u} \end{aligned} \quad (17)$$

where in the last two equalities we used the Nishimori identity (10). By a similar calculation,

$$\begin{aligned} \mathbb{E} \langle L(\underline{x}) \rangle_{t,u} \mathbb{E} \langle q_{12} \rangle_{t,u} &= \frac{1}{2} \mathbb{E} \langle 1 - q_{12} + 2m_1 \rangle_{t,u} \mathbb{E} \langle q_{12} \rangle_{t,u} \\ &= \frac{1}{2} (\mathbb{E} \langle m_1 \rangle_t + (\mathbb{E} \langle m_1 \rangle_t)^2) \end{aligned} \quad (18)$$

From equations (17) and (18), we get

$$\int_{\epsilon}^a du \mathbb{E} |\langle m_1^2 \rangle_{t,u} - (\mathbb{E} \langle m_1 \rangle_{t,u})^2| \leq O\left(\frac{1}{K^{\frac{1}{8}}}\right)$$

Now integrating with respect to t and exchanging the integrals (by Fubini's theorem), we get

$$\int_{\epsilon}^a du \int_0^1 dt \mathbb{E} |\langle m_1^2 \rangle_{t,u} - (\mathbb{E} \langle m_1 \rangle_{t,u})^2| \leq O\left(\frac{1}{K^{\frac{1}{8}}}\right)$$

The limit of the left hand side as $K \rightarrow \infty$ therefore vanishes. By Lebesgue's theorem this limit can be exchanged with the u integral and we get the desired result. (Note that one can further exchange the limit with the t -integral and obtain that the fluctuations of m_1 vanish for almost every (t, u)). ■

V. EXTENSIONS

In this section we briefly describe two variations for which our methods extend in a straightforward manner.

A. Unequal Powers

The above method can be applied to the case where the users have unequal powers. Let P_k be the power used by user k and let the average power be normalized $\frac{1}{K} \sum P_k = 1$. We assume that the empirical distribution of these powers tends to a distribution and denote the corresponding expectation by $\mathbb{E}_P[-]$.

The correct "order parameters" in this case are $m_1 = \frac{1}{N} \sum P_k x_i$ and $q_{12} = \frac{1}{N} \sum P_k x_k^{(1)} x_k^{(2)}$. Applying the interpolation method as in the previous section yields

$$\begin{aligned} \mathcal{F}(1) &= -\frac{1}{2\beta} + \mathbb{E}_P \int Dz \ln(2 \cosh(\sqrt{PF} z + PF)) \\ &\quad - F(1 + m)/2 - \frac{1}{2\beta} \ln(1 + \beta B(1 - m)) - \frac{1}{2} + \int R(t) dt \end{aligned}$$

where $R(t)$ has the same form as before but the with new definition of m_1 .

From the positivity of $R(t)$ we deduce that the replica solution gives an upper bound to the capacity.

B. Gaussian Input

The same method can be used when the input is Gaussian. Here we cannot assume the input sequence x_k^0 to be the all 1 sequence. However this is not a problem as this assumption was just done for convenience in the binary input case. The necessary change are the following: a) one replaces z_k by $x_k^0 - x_k$; b) the expectation \mathbb{E} includes the expectation over a gaussian for the input \underline{x}^0 ; c) the Gibbs averages $\langle - \rangle_t$ have an extra prior distribution for the continuous gaussian variables \underline{x} and the sums in the normalization factors Z_t are replaced by integrals; d) the order parameters are $m_1 = \frac{1}{K} \sum x_k^0 x_k$ and $q_{12} = \frac{1}{K} \sum x_k^{(1)} x_k^{(2)}$.

The interpolation method then yields

$$\mathcal{F}(1) = -\frac{1}{2\beta} - \frac{1}{2} \ln(1 + F) - \frac{1}{2} - \frac{1}{2\beta} \ln(1 + \beta B(1 - m)) \\ + F(1 - m)/2 + \int R(t) dt$$

where $R(t)$ is the same function as before but with new definition of m_1 . Again the positivity of $R(t)$ implies that the replica solution is an upper bound to the capacity. Of course the formula matches that of RMT analysis [7], [2].

Remark 2: The method can be adapted to a wider class of input distributions.

VI. BINARY SPREADING SEQUENCES

In this section we report on some progress concerning the case of binary spreading sequence. There are two main issues. First the concentration of capacity and free energy, and second the proof of the replica formula (3).

Let us first comment on the second issue that we hope to solve in a future contribution. From replica calculations it is conjectured that the same replica formula holds independently of the nature of the spreading sequence as long as they are i.i.d and symmetric random variables. It is not completely clear how to adapt the calculations of section III because we lack a nice gaussian integration by parts formula (however it is possible to devise a similar but more cumbersome formula for binary variables). However we think that there should be a direct way of showing the independence by “interpolating” between the binary and gaussian cases.

For the moment we have solved the first issue and present a proof of concentration for the capacity and free energy in the specific case of $s_{ik} \sim \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. The results here are weaker than in the gaussian case because we get only power law concentration. However the proof of this case is much more general and applies almost verbatim to all symmetric distributions of s_{ik} with finite fourth moment. Let us first state the main results.

Following ideas from [9], [10] we use martingale arguments to show that the fluctuations of the capacity go to zero.

Lemma 6: $\mathbb{E}_{\mathbf{S}}[(C(\mathbf{S}) - \mathbb{E}_{\mathbf{S}}[C(\mathbf{S})])^2] = O\left(\frac{1}{K}\right)$

The proof of this lemma is given at the end of this section. Using Chebyshev’s inequality we deduce

Theorem 5: [concentration of Capacity] There exists an integer K_1 , such that for all $K > K_1$

$$\mathbb{P}(|C(\mathbf{S}) - \mathbb{E}_{\mathbf{S}}[C(\mathbf{S})]| \geq \epsilon) = O\left(\frac{1}{K\epsilon^2}\right)$$

Concerning the free energy the situation is more complicated because one has to show concentration with respect to both the gaussian noise and the binary spreading sequence. The proofs combine the method of the present section with those used for gaussian spreading sequences in [8]. Since the analysis is quite lengthy we just give the main result

Theorem 6: [concentration of free energy] There exists an integer K_1 , such that for all $K > K_1$ and all $\epsilon > 0$

$$\mathbb{P}(f(\underline{Y}, \mathbf{S}) - \mathbb{E}_{\underline{Y}, \mathbf{S}}[f(\underline{Y}, \mathbf{S})] \geq \epsilon) = O\left(\frac{1}{K\epsilon^2}\right)$$

The rest of this section is devoted to the proof of lemma 6. Here, for simplicity we assume the noise variance to be 1.

Proof of lemma 6: For $l \leq K$, let ϕ_l be the sigma algebra generated by $\{s_{ik} : 1 \leq i \leq N, 1 \leq k \leq l\}$. and set

$$f_l = \mathbb{E}[C(\mathbf{S}) | \phi_l], \quad \psi_l = f_l - f_{l-1}$$

Then

$$\mathbb{E}(C(\mathbf{S}) - \mathbb{E}[C(\mathbf{S})])^2 = \sum_{l=1}^K \mathbb{E}[\psi_l^2]$$

The goal is to bound each term in this sum by $O(\frac{1}{K^2})$. As discussed in section III, for a particular realization of \mathbf{S} we cannot assume the all one sequence being transmitted. Hence we use the following form of the capacity.

$$C(\mathbf{S}) = \ln 2 - \frac{1}{2\beta} - \mathbb{E}_{\underline{x}} \left[\sum_{\underline{x}^0} \frac{1}{2^K} \ln \sum_{\underline{x}} e^{H(\underline{x}_0, \underline{x})} \right]$$

where,

$$H(\underline{x}_0, \underline{x}) = -\frac{1}{2} \sum_i \left(n_i + \frac{1}{\sqrt{N}} \sum_k s_{ik} (x_k^0 - x_k) \right)^2$$

From now on in the notation, we do not explicitly show the dependency of H on \underline{x}^0 and \underline{x} . To this end we define the following three Hamiltonians.

$$H_l = \frac{-1}{2N} \sum_{k_1, k_2 \neq l, i} s_{ik_1} s_{ik_2} (x_{k_1}^0 - x_{k_1})(x_{k_2}^0 - x_{k_2}) \\ + \frac{1}{\sqrt{N}} \sum_{i, k \neq l} n_i s_{ik} x_k$$

$$R_l = \frac{1}{2N} \sum_i s_{il}^2 (x_l^0 - x_l)^2 \\ - \frac{1}{N} \sum_{i, k} s_{ik} s_{il} (x_l^0 - x_l)(x_k^0 - x_k) + \frac{1}{\sqrt{N}} \sum_i n_i s_{il} x_l$$

$$\tilde{H}_l(t) = H_l + tR_l$$

where $t \in [0, 1]$ will play the role of an interpolating parameter. We also introduce the difference of free energies associated to the hamiltonian $\tilde{H}_l(t)$ and H_l ,

$$\tilde{f}_l(t) = \frac{1}{2^K} \sum_{\underline{x}^0} (\ln Z(\tilde{H}_l(t)) - \ln Z(\tilde{H}_l(0)))$$

In the last definition the partition function is defined by the usual summation over all configurations \underline{x} .

With these definitions we have the representation

$$\psi_l = \frac{1}{K} \mathbb{E}_{\geq l+1} \tilde{f}_l(1) - \frac{1}{K} \mathbb{E}_{\geq l} \tilde{f}_l(1)$$

where $\mathbb{E}_{\geq l}$ means expectation with respect to $\{s_{ik} \forall k \geq l\}$. It follows that

$$\begin{aligned} \mathbb{E}[\psi_l^2] &\leq \frac{1}{K^2} \mathbb{E} \mathbb{E}_{\geq l+1} \tilde{f}_l(1)^2 + \frac{1}{K^2} \mathbb{E} \mathbb{E}_{\geq l} \tilde{f}_l(1)^2 \\ &\quad - \frac{2}{K^2} \mathbb{E}[(\mathbb{E}_{\geq l+1} \tilde{f}_l(1) | \phi_{l-1})(\mathbb{E}_{\geq l} \tilde{f}_l(1))] \\ &= \frac{2}{K^2} \mathbb{E} \tilde{f}_l(1)^2 - \frac{2}{K^2} \mathbb{E}[(\mathbb{E}_{\geq l} \tilde{f}_l(1))^2] \\ &\leq \frac{2}{K^2} \mathbb{E} \tilde{f}_l(1)^2 \end{aligned}$$

Notice that $\tilde{f}_l(0) = 0$ and $\frac{d^2}{dt^2} \tilde{f}_l(t) \geq 0$. Therefore,

$$\tilde{f}_l'(0) \leq \tilde{f}_l(1) \leq \tilde{f}_l'(1)$$

and

$$\mathbb{E}[\tilde{f}_l(1)^2] \leq \mathbb{E}[\tilde{f}_l'(0)^2] + \mathbb{E}[\tilde{f}_l'(1)^2]$$

This shows that our task is reduced to a proof of $\mathbb{E}[\tilde{f}_l'(0)^2] = O(1)$, $\mathbb{E}[\tilde{f}_l'(1)^2] = O(1)$. This is a technical calculation which we omit due to lack of space. ■

APPENDIX

A. Nishimori Identities

The channel symmetry implies a set of powerful Nishimori identities. The reason is that it induces a gauge symmetry relating the averages of various observables.

We define the distributions of m_1 and q_{12} as

$$\mathbb{P}_{m_1}^t(u) = \mathbb{E} \langle \delta(m_1 - u) \rangle_t, \quad \mathbb{P}_{q_{12}}^t(u) = \mathbb{E} \langle \delta(q_{12} - u) \rangle_t$$

where it is understood that we assume the input sequence to be the all 1.

$$\text{Lemma 7: } \mathbb{P}_{m_1}^t(u) = \mathbb{P}_{q_{12}}^t(u)$$

Proof: We only give a brief sketch because the method is standard (see for example [11]). One writes fully explicitly the expression for $\mathbb{P}_{m_1}^t(u)$ (as defined above) and performs the gauge transformation $x_k \rightarrow x_k^0 x_k$, $s_{ik} \rightarrow x_k^0 s_{ik}$ where \underline{x}^0 is an arbitrary binary sequence. Since $\mathbb{P}_{m_1}^t(u)$ does not depend on \underline{x}^0 we sum over all such 2^K sequences and obtain a lengthy expression. Exactly the same procedure is applied to $\mathbb{P}_{q_{12}}^t(u)$ and one gets another lengthy expression. Then one can recognize that these two expressions are the same. ■

Let X_i and X_{11} be defined as before, then

$$\text{Lemma 8: } \mathbb{E} \langle X_{11} \rangle_t = 1$$

Proof: The proof uses gauge transformation and is similar to the proof of lemma 7. ■

Lemma 9:

$$\begin{aligned} \sum_{i,k} \mathbb{E} \langle n_i(n_i + \sqrt{\frac{B(t)}{N}} \sum_l s_{il} z_l^{(2)}) z_k^{(1)} z_k^{(2)} \rangle_t \\ = \sum_{i,k} \mathbb{E} \langle n_i(n_i + \sqrt{\frac{B(t)}{N}} \sum_l s_{il} z_l) z_k \rangle_t \end{aligned}$$

Proof: The proof uses gauge transformations and is similar to the proof of lemma 7. ■

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